

Feedback classification of linear systems over von Neumann regular rings

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Abstract

It is proved that feedback classification of a linear system over a commutative von Neumann regular ring R can be reduced to the classification of a finite family of systems, each of which is properly split into a reachable and a non-reachable part, where the reachable part is in a Brunovski-type canonical form, while the non-reachable part can only be altered by similarity. If a canonical form is known for similarity of matrices over R , then it can be used to construct a canonical form for feedback equivalence. An explicit algorithm is given to obtain the canonical form in a computable context together with an example over a finite ring.

1 Introduction and motivation

Let R be a commutative ring with 1. An m -input, n -dimensional linear system over R is a pair of matrices (A, B) , with $A \in R^{n \times n}$, $B \in R^{n \times m}$. The control theoretic background for this terminology is nicely described in [1, Introduction]. Within this context two systems (A, B) and (A', B') are called *feedback equivalent* (*f.e.*, $\stackrel{f.e.}{\sim}$) if there exist matrices P, Q, K of suitable sizes such that $(A', B') = (P(A + BK)P^{-1}, PBQ)$, with P, Q invertible.

Note that equivalence and similarity of matrices over R are particular cases of feedback equivalence, indeed, if (A, B) and (A', B') are feedback equivalent, then B, B' are equivalent, while the equivalence of $(A, 0), (A', 0)$ implies similarity of A, A' .

Classification and canonical forms under feedback equivalence for linear systems are classical problems in linear control theory. While a general solution for arbitrary rings can not be expected several specific results have been obtained for special classes of commutative rings. If R is a field, controllability indices and Brunovski canonical form for reachable systems together with Kalman's decomposition give a complete description.

For larger classes of rings a variety of partial results have been obtained so far, e.g. a partial canonical form for reachable systems over principal ideal domains [3] or a canonical form for weakly reachable single-input systems over Bézout domains [5].

In [4], [9] it was observed, that certain basic control problems have a solution if and only if the underlying ring is regular (in the sense of von Neumann, see definition below):

- local and global feedback equivalence over R are equivalent iff R is regular.
- cyclizability of the reachable part of systems is always possible iff R is regular.

For details see [4, Theorem 4.1] and [9, Theorem 4].

At the same time regular rings have nice linear algebraic properties, for example:

- $R^{n \times n}$ is regular iff R is regular [6].
- The tensor criterion for matrix similarity of Byrnes and Gauger is valid iff the underlying ring is regular [7].

This encourages to study feedback classification and canonical forms over the large class of commutative regular rings. A commutative ring R is said to be (von Neumann) *regular* if for any a in R there exists $x \in R$ such that $a^2x = a$.

The following mostly well-known properties of regular rings will be of use for us later, see [6] for details.

- Any finitely generated ideal is principal (i.e. R is a Bézout ring) and generated by an idempotent.
- Any element is the product of a unit with an idempotent.
- R has stable range 1, which for a Bézout ring means that whenever $aR + bR = dR$ for some $a, b, d \in R$ then there exists $c \in R$ such that $a + bc = ud$, with u a unit.
- R is an elementary divisor ring, which means that there exists a ‘Smith Form’: every matrix B is equivalent to a diagonal matrix with diagonal entries $d_1|d_2|\dots|d_r$. In fact, the Smith Form is unique, if the d_i ’s are idempotents (see Lemma 4 below).
- Any finitely generated submodule of R^n has a finitely generated complement.

Note also that any field is a regular ring and more generally any subring of any product of fields and in particular many rings of functions comprising many finite rings.

2 Preliminaries

For an n -dimensional system $\Sigma = (A, B)$ over a commutative ring the right image \mathcal{R}_Σ of $[B, AB, \dots, A^{n-1}B]$ is called the *module of reachable states*. Σ is *reachable* if $\mathcal{R}_\Sigma = R^n$. When R is regular, then \mathcal{R}_Σ is always a (finitely generated) direct summand of R^n with a finitely generated complement. If \mathcal{R}_Σ has a finite basis which can be completed to a basis of R^n , then the system Σ is f.e. to a system of the form

$$\left(\begin{bmatrix} A' & A'' \\ 0 & A''' \end{bmatrix}, \begin{bmatrix} B' \\ 0 \end{bmatrix} \right)$$

where (A', B') is reachable. This is usually called a *Kalman decomposition*, which we shall call *strong* in case $A'' = 0$. The reachable system (A, B) is in so called *Brunovski form* if

$$A = \text{diag}(A_1, \dots, A_s), B = [\text{diag}(B_1, \dots, B_s), 0]$$

where $A_i = [e^{(2)}, \dots, e^{(n_i)}, 0]$, $B_i = e^{(1)}$ with $e^{(j)}$ the j -th canonical basis vector in R^{n_i} for $1 \leq i \leq s$ and where $n_1 \geq \dots \geq n_s$, $n_1 + \dots + n_s = n$. The n_i are called controllability or also Kronecker indices. They are a complete set of invariants under feedback in case R is a field. See e.g. [10, chapters 3.3 and 4.2]

Typically regular rings contain many idempotent elements. Therefore we now collect some of their basic properties.

Lemma 1 *Let R be a commutative ring and e_1, \dots, e_k pairwise orthogonal idempotents ($e_i e_j = 0$ if $i \neq j$) and such that $\sum_{i=1}^k e_i = 1$. Then:*

$$R = e_1 R \oplus \dots \oplus e_k R$$

Note that the ideal $e_i R$ is at the same time a regular ring with e_i as unit element.

An immediate consequence of such a decomposition is, that any matricial equation $M = M'$ holds over R if and only if $e_i M = e_i M'$ holds for all i , in particular:

Lemma 2 - *A matrix P is invertible over R iff $e_i P$ is invertible over $e_i R$ for all i .*
- *Two matrices A, A' are similar (resp. equivalent) over R iff $e_i A, e_i A'$ are similar (resp. equivalent) over $e_i R$ for all i .*
- *Two systems (A, B) and (A', B') are f.e. over R iff $(e_i A, e_i B)$ and $(e_i A', e_i B')$ are f.e. over $e_i R$ for all i .*

Next, we recall basic facts about feedback equivalence.

Lemma 3 *Let R be a commutative ring with 1, and consider the system (A, B) over R of size $(n, m), 0 < m < n$ given by*

$$A = \begin{bmatrix} 0 & 0 \\ B_1 & A_1 \end{bmatrix}, B = \begin{bmatrix} I \\ 0 \end{bmatrix},$$

with $I \in R^{m \times m}$ an identity matrix, $A_1 \in R^{(n-m) \times (n-m)}$ and the remaining blocks of appropriate sizes. Then, one has:

(i) *The feedback equivalence class of (A, B) is uniquely determined by the feedback equivalence class of the $(n-m, m)$ system (A_1, B_1) in the following sense: two systems*

$$\left(\begin{bmatrix} 0 & 0 \\ B_1 & A_1 \end{bmatrix}, \begin{bmatrix} I \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} 0 & 0 \\ B'_1 & A'_1 \end{bmatrix}, \begin{bmatrix} I \\ 0 \end{bmatrix} \right)$$

are equivalent if and only if $(A_1, B_1), (A'_1, B'_1)$ are equivalent.

(ii) *If R is a principal ideal domain or has stable range 1 (in particular, if R is regular), (i) is also valid if B has any number of additional zero columns, i.e. it has the block form:*

$$B = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

(iii) *If (A_1, B_1) is a reachable system in Brunovski canonical form then also (A, B) is reachable and can be transformed just by permutation of rows and columns, to Brunovski form. Moreover the controllability indices $\kappa_1 \geq \dots \geq \kappa_m, \lambda_1 \geq \dots \geq \lambda_m$ (possibly some of the κ_i have to be put to zero) of (A_1, B_1) and (A, B) are related by $\lambda_i = \kappa_i + 1$ for $1 \leq i \leq m$.*

Proof. (i) The proof is straightforward and can be adapted from [5, Lemmas 2.1 and 2.2].

(ii) See [8, Propositions 2.4 and 2.5].

(iii) When (A_1, B_1) is in Brunovski form, then for some s we have

$$Ae^{(1)} = e^{(m+1)}, A^2e^{(1)} = e^{(m+2)}, \dots, A^se^{(1)} = e^{(m+s)}.$$

Similar relations are valid for $e^{(2)}, \dots, e^{(m)}$. Thus an appropriate renumbering of the canonical basis vectors gives a Brunovski form and the relation for the indices is straightforward.

□

3 Feedback classification and canonical forms

Let R be a regular ring, and consider a system $\Sigma = (A, B)$ of size (n, m) over R . We will always tacitely assume $B \neq 0$. In the sequel we will construct recursively a normal form and a complete set of invariants for the feedback classification of Σ . The following reduction step will be essential:

Lemma 4 *By operations of the feedback group any system (A, B) can be assumed to have the following form*

$$A = (a_{ij})_{i,j=1}^n, \quad B = \left[\begin{array}{ccc|c} d_1 & & & 0 \\ & \ddots & & \\ & & d_r & \\ \hline & 0 & & 0 \end{array} \right] \quad (1)$$

with $d_1 | d_2 | \dots | d_r$, where all the d_i 's are nonzero idempotents and where for $i = 1, \dots, r$, the i -th row of A is orthogonal to d_i . Furthermore, the elements d_1, \dots, d_r are invariant under feedback equivalence. If $r = n$ or $r = m$ the corresponding 0-blocks will not occur.

Proof. Since R is an elementary divisor ring and any element of R is a product of an idempotent with a unit, the form of B is obtained via matrix equivalence. Moreover, if we denote by a_{ij} the element in position (i, j) of A , with a suitable feedback action it can be replaced by $a_{ij} - d_i a_{ij} = (1 - d_i) a_{ij}$, which is orthogonal to d_i .

For each $i = 1, \dots, r$, the ideal of R generated by all the $i \times i$ minors of B is invariant under equivalence and thus invariant under feedback equivalence of the system (A, B) . But this ideal is clearly generated by $d_1 \cdots d_i = d_i$, and an ideal in R cannot be generated by two distinct idempotents. \square

We are now ready to solve the feedback classification problem for systems over a von Neumann regular ring R . A first attempt in this direction could be to reduce the problem to the classification of systems over all residue fields (see [4, Theorem 4.1]): two systems over R are equivalent if and only if they are equivalent over the residue field R/\mathfrak{m} , for all maximal ideals \mathfrak{m} . However, there may exist an infinite number of maximal ideals in the regular ring R . But, as we will prove in the next theorem, the feedback classification problem of a system over R can be reduced to the classification of a *finite* family of systems which *behave like* systems over fields.

Theorem 5 (Canonical forms and invariants) *Let $\Sigma = (A, B)$ be a system of size (n, m) over a regular ring R . Then:*

- (i) *There exists a finite family of idempotents $\{e_k\}$, pairwise orthogonal and with sum 1, such that for each k , the system $(e_k A, e_k B)$ over the regular ring $e_k R$ is feedback equivalent to a system in strong Kalman decomposition, with the block form*

$$\left(\begin{bmatrix} \widehat{A}_k & 0 \\ 0 & \widehat{C}_k \end{bmatrix}, \begin{bmatrix} \widehat{B}_k \\ 0 \end{bmatrix} \right),$$

where the pair $(\widehat{A}_k, \widehat{B}_k)$ is reachable and in Brunovski canonical form over $e_k R$.

- (ii) *With these notations, a complete set of feedback invariants for (A, B) consists of:*

- The idempotents $\{e_k\}$, which can be obtained successively from the invariant factors of a finite number of matrix equivalences.
- The similarity classes of the matrices \widehat{C}_k .
- The controllability indices of the systems $(\widehat{A}_k, \widehat{B}_k)$.

Proof. We will proceed by induction on n .

• Proof of (i) and (ii) in the case $n = 1$. We can assume without loss of generality that the system Σ is given in the form (1): $A = [a], B = [d, 0 \cdots 0]$, with d idempotent and $ad = 0$. Let us consider the partition $R = dR \oplus (1 - d)R$, and build the systems:

$$(1 - d)\Sigma = ([(1 - d)a], [0 \cdots 0]) \quad \text{and} \quad d\Sigma = ([0], [d, 0 \cdots 0]) = d([0], [1, 0, \dots, 0])$$

It is clear that both systems are decomposed in strong Kalman form, the first one without reachable part, and the second one already in Brunovski form over the ring dR , in which d is the 1-element. Moreover, the element d is obtained from the equivalence of B and is invariant by Lemma 4, making $1 - d$ also invariant. Now, by Lemma 2, the feedback equivalence class of Σ is completely determined by that of $d\Sigma$ (with a single invariant d), and that of $(1 - d)\Sigma$, where the first entry $(1 - d)a$ in principle can be reduced by similarity, but in the 1×1 -case this has no effect.

• Proof of (i) for $n > 1$. Let Σ be given as in (1), and consider the following family of idempotents (pairwise orthogonal and with sum 1):

$$\begin{cases} e_0 &= 1 - d_1 \\ e_1 &= d_1(1 - d_2) \\ &\vdots \\ e_{r-1} &= d_{r-1}(1 - d_r) \\ e_r &= d_r \end{cases}$$

For $i = 0$, we have that $e_0 = 1 - d_1$ is orthogonal to d_1 , and hence also orthogonal to all the remaining d_i 's, so that $e_0 B = 0$, which means that $e_0 \Sigma$ trivially is a strong Kalman decomposition with no reachable part. This step is omitted if $d_1 = 1$.

For $i = 1, \dots, \min\{r, n - 1\}$, note that e_i is a multiple of d_1, \dots, d_i and orthogonal to d_{i+1}, \dots, d_r and to the rows $1, \dots, i$ of A . Note also, that $e_i = 0$ if $d_i = d_{i+1}$.

Therefore we obtain for $1 \leq i \leq \min\{r, n - 1\}$ and if $e_i \neq 0$:

$$e_i \Sigma = (e_i A, e_i B) = \left(\left[\begin{array}{c|c} 0 & 0 \\ \hline e_i B_i & e_i A_i \end{array} \right], \left[\begin{array}{c|c} e_i I & 0 \\ \hline 0 & 0 \end{array} \right] \right)$$

with $A_i \in R^{(n-i) \times (n-i)}$, I an $i \times i$ identity block, and the remaining blocks of appropriate sizes.

Since $(e_i A_i, e_i B_i)$ is of size $(n - i, i)$ (with $n - i < n$), by the induction assumption one can assume that there exists a finite partition $e_i R = \bigoplus_j e_{ij} R$, such that for each j , the system $(e_{ij} A_i, e_{ij} B_i)$ over the ring $e_{ij} R$ is equivalent to one of the form

$$\left(\left[\begin{array}{c|c} \widetilde{A}_{ij} & 0 \\ \hline 0 & \widetilde{C}_{ij} \end{array} \right], \left[\begin{array}{c} \widetilde{B}_{ij} \\ 0 \end{array} \right] \right),$$

with $(\widetilde{A}_{ij}, \widetilde{B}_{ij})$ reachable and in Brunovski canonical form.

In virtue of Lemma 3,(i),(ii), we obtain

$$e_{ij}\Sigma = (e_{ij}A, e_{ij}B) \stackrel{f.e.}{\sim} \left(\left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline \widetilde{B}_{ij} & \widetilde{A}_{ij} & 0 \\ 0 & 0 & \widetilde{C}_{ij} \end{array} \right], \left[\begin{array}{c|c} e_{ij}I & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array} \right] \right)$$

The system on the right side can also be written as

$$\left(\left[\begin{array}{cc|c} 0 & 0 & 0 \\ \hline \widetilde{B}_{ij} & \widetilde{A}_{ij} & 0 \\ 0 & 0 & \widetilde{C}_{ij} \end{array} \right], \left[\begin{array}{cc} e_{ij}I & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \right] \right).$$

By Lemma 3,(iii) we know that

$$\left(\left[\begin{array}{cc} 0 & 0 \\ \hline \widetilde{B}_{ij} & \widetilde{A}_{ij} \end{array} \right], \left[\begin{array}{cc} e_{ij}I & 0 \\ 0 & 0 \end{array} \right] \right) \stackrel{f.e.}{\sim} (\widehat{A}_{ij}, \widehat{B}_{ij}),$$

with $(\widehat{A}_{ij}, \widehat{B}_{ij})$ in Brunovski form. If we let $\widehat{C}_{ij} = \widetilde{C}_{ij}$, then we obtain finally

$$e_{ij}\Sigma \stackrel{f.e.}{\sim} \left(\left[\begin{array}{cc} \widehat{A}_{ij} & 0 \\ 0 & \widehat{C}_{ij} \end{array} \right], \left[\begin{array}{c} \widehat{B}_{ij} \\ 0 \end{array} \right] \right)$$

with the desired properties for the righthand side system. This completes the study of the systems $e_i\Sigma$, for $i = 1, \dots, \min\{r, n-1\}$.

If $r = n$, the system $e_n\Sigma$ is already in Brunovski form, because e_nA is zero and $e_nB = [e_nI|0]$, with I an $n \times n$ identity block.

To sum up, we have obtained a finite partition of R with idempotents $\{e_{ij}\}$ such that for each i, j , the system $e_{ij}\Sigma$ is a strong Kalman decomposition with reachable part in Brunovski form, as required.

- Proof of (ii) for $n > 1$. With the preceeding notations, the elements $\{e_i\}$ are constructed from the Smith form of B and satisfy the conditions of Lemma 1, from which it follows by Lemma 2 that the classification of Σ over R is reduced to the classification of the systems $e_i\Sigma$ over the regular rings e_iR (at this step we need only consider those indices i for which $e_i \neq 0$). A complete set of invariants for Σ will be given by the elements e_0, \dots, e_r , together with all the invariants obtained recursively.

For $i = 0$ we have seen in (i) that $e_0B = 0$, i.e. the feedback class of (e_0A, e_0B) is reduced to the similarity class of e_0A .

For $i = 1, \dots, \min\{r, n-1\}$, by Lemma 3 the feedback class of $e_i\Sigma$ is completely determined by the feedback class of the system (e_iA_i, e_iB_i) , which by the induction hypothesis (see the notation in the proof of part (i)) reduces to a collection of idempotents $\{e_{ij}\}$ obtained from equivalences of matrices, the similarity class of all matrices \widetilde{C}_{ij} , and the controllability indices of all the systems $(\widetilde{A}_{ij}, \widetilde{B}_{ij})$, which are directly related to the controllability indices of the augmented systems $(\widehat{A}_{ij}, \widehat{B}_{ij})$ as was explained in Lemma 3,(iii).

Finally, if $r = n$, the additional system $e_n\Sigma$ has the trivial form $(0, [e_nI|0])$ and does not provide any new invariant. This completes the proof. \square

4 The reachable and the single input case

As an immediate consequence, reachable systems can be completely described without reference to similarity, i.e. only with idempotents and Brunovski blocks.

Corollary 6 (Reachable systems) *With the above notations, every reachable system over a regular ring is equivalent to a direct sum of systems in Brunovski canonical form, and the feedback equivalence class can be described by a family of idempotents and Brunovski indices.*

Proof. With the notations of the previous theorem, reachability of a given system (A, B) is equivalent to reachability of all the systems

$$(e_k A, e_k B) \stackrel{f.e.}{\sim} \left(\begin{bmatrix} \widehat{A}_k & 0 \\ 0 & \widehat{C}_k \end{bmatrix}, \begin{bmatrix} \widehat{B}_k \\ 0 \end{bmatrix} \right),$$

which is only possible if none of the blocks \widehat{C}_k appear, i.e. one can remove all references to similarity in the previous theorem. \square

The description via a finite collection of idempotents seems essential and unavoidable for an explicit exposure of some kind of normal form. Nevertheless, as in the classical field case, over a regular ring it is possible to classify reachable linear systems in a more global, but at the same time more abstract way, by the following sequence of submodules of the module \mathcal{R}_Σ of reachable states.

For an n -dimensional system $\Sigma = (A, B)$ over a regular ring R let

$$N_k^\Sigma = \text{im}[B|AB|\cdots|A^{k-1}B],$$

for $k = 1, \dots, n$.

Proposition 7 (Reachable systems and the modules N_k^Σ) *Let $\Sigma = (A, B)$ and $\Sigma' = (A', B')$ be two reachable systems of size (n, m) over a regular ring R . Then, the following statements are equivalent:*

- (i) Σ and Σ' are feedback equivalent.
- (ii) The R^n -submodules N_k^Σ and $N_k^{\Sigma'}$ are isomorphic for all $k = 1, \dots, n$.

Proof. (i) \Rightarrow (ii) is trivial. (ii) \Rightarrow (i) can be derived directly from results in [4]. Let \mathfrak{m} be an arbitrary maximal ideal of R . If we denote by $\Sigma(\mathfrak{m})$ and $\Sigma'(\mathfrak{m})$ the extensions of Σ, Σ' to the residue field R/\mathfrak{m} (by reduction modulo \mathfrak{m}), since $N_k^\Sigma \cong N_k^{\Sigma'}$ we must have

$$N_k^{\Sigma(\mathfrak{m})} \cong N_k^{\Sigma'(\mathfrak{m})}$$

for $k = 1, \dots, n$. But the dimensions of the R/\mathfrak{m} -vector spaces $N_k^{\Sigma(\mathfrak{m})}, N_k^{\Sigma'(\mathfrak{m})}$ characterize completely the equivalence classes of the reachable systems $\Sigma(\mathfrak{m}), \Sigma'(\mathfrak{m})$ over the field R/\mathfrak{m} (see [4, Proposition 2.5]), therefore it follows that $\Sigma(\mathfrak{m}) \stackrel{f.e.}{\sim} \Sigma'(\mathfrak{m})$ over R/\mathfrak{m} . Now, by [4, Theorem 4.1] we can conclude that $\Sigma \stackrel{f.e.}{\sim} \Sigma'$ over the regular ring R . \square

Proposition 7 is yet another instance where regular rings behave “classically”. Even for principal ideal domains a corresponding general result is not possible because of a counterexample in [2, section 6.2].

As a consequence of Proposition 7 we obtain

Corollary 8 *The feedback equivalence class of a reachable system $\Sigma = (A, B)$ of size (n, m) over a regular ring R is completely determined by the invariant factors of the matrices $[B|AB|\cdots|A^{k-1}B]$, for $k = 1, \dots, n$.*

Proof. The result is clear, since for each $k = 1, \dots, n$, the columns of the matrix $[B|AB|\cdots|A^{k-1}B]$ generate the R -module N_k^Σ , whose isomorphism class is determined by the invariant factors of the generating matrix. \square

In the single-input case, the reachable part of a system can be transformed to a generalized controller canonical form. All notations are as before.

Proposition 9 (Single input systems) .

(i) *A canonical form for a system $\Sigma = (A, B)$, $B \neq 0$, of size $(n, 1)$ over a regular ring R is given by*

$$\tilde{A} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ d_2 & 0 & \ddots & \vdots & \vdots \\ 0 & d_3 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & d_n & 0 \end{bmatrix} + A^*, \quad \tilde{B} = \begin{bmatrix} d_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

with idempotents $d_1|d_2|\cdots|d_n$, possibly $d_{r+1} = \cdots = d_n = 0$ for some $r \geq 1$, and with

$$A^* = \tilde{A}_{1-d_1} + \begin{bmatrix} 0 & 0 \\ 0 & \tilde{A}_{d_1(1-d_2)} \end{bmatrix} + \cdots + \begin{bmatrix} 0 & 0 \\ 0 & \tilde{A}_{d_{r-1}(1-d_r)} \end{bmatrix},$$

where $\tilde{A}_{d_i(1-d_{i+1})} \in R^{(n-i) \times (n-i)}$ for $1 \leq i < r$ and $d_{i+1}\tilde{A}_{d_i(1-d_{i+1})} = 0$ for $0 \leq i < r$.

(ii) *For $k = 1, \dots, n$ one has*

$$N_k^{(A,B)} \cong N_k^{(\tilde{A}, \tilde{B})} = d_1 e^{(1)} R + \cdots + d_k e^{(k)} R \cong d_1 R \oplus d_2 R \oplus \cdots \oplus d_k R$$

(iii) *The elements d_1, \dots, d_n can be obtained as the idempotent invariant factors appearing in the Smith form of the matrix $[B, AB, \dots, A^{n-1}B]$.*

Proof. 1.: For a system $\Sigma = (A, B)$ of size $(n, 1)$, the reduced form (1) yields only one idempotent invariant factor d_1 , i.e. two idempotents $e_0 = 1 - d_1$ and $e_1 = d_1$ as in the proof of Theorem 5. The system $e_0 \Sigma$ can only be reduced further by similarity and is characterized by some corresponding canonical form \tilde{A}_{1-d_1} , while the class of $d_1 \Sigma$ is reduced to that of a system $(d_1 A_1, d_1 B_1)$ of size $(n-1, 1)$. Continuing this iteration r times, at each step i we will have a system $(d_i A_i, d_i B_i)$ of size $(n-i, 1)$, with d_{i+1} the idempotent invariant factor of $d_i B_i$ ($d_i|d_{i+1}$), and $\tilde{A}_{d_i(1-d_{i+1})}$ a canonical form of $d_i(1-d_{i+1})A_i$ for similarity over the ring $d_i(1-d_{i+1})R$. Multiple use of Lemma 3,(i) leads to the stated result.

2. and 3.: The modules N_k^Σ are invariant under feedback and remain isomorphic under feedback equivalence. Therefore 2. and 3. are straightforward consequences of 1. \square

5 Implementation

The procedure given in Theorem 5 is constructive, which gives rise to an explicit algorithm, provided Smith normal forms are computable over the ring R , and a canonical form is known for the similarity relation.

Algorithm 10 (Canonical forms)

INPUT: (e, A, B) , with (A, B) a system and e an idempotent (initially $e = 1$)

OUTPUT: A list of lists $[e_k, \widehat{A}_k, \widehat{B}_k]$, where e_k is an idempotent and $(\widehat{A}_k, \widehat{B}_k)$ is a canonical form of (A, B) over the ring $e_k R$.

1. Replace (A, B) by its reduced form (1), with $d_1 | d_2 | \dots | d_r$ idempotents.
2. Initialize $L = []$ (this will be the output).
3. Define $\{e_0 = e - d_1, e_1 = d_1(e - d_2), \dots, e_{r-1} = d_{r-1}(e - d_r), e_r = d_r\}$.
4. For each $e_i \neq 0$, do steps 5–8.
5. If $e_i B = 0$, add $[e_i, \widehat{A}_i, 0]$ to L , where \widehat{A}_i is a canonical form of A for similarity over $e_i R$. Proceed with next e_i .
6. If $i = r = n$, add $[e_i, 0, e_i B]$ to L .
7. Extract blocks $e_i A_i, e_i B_i$ from $e_i A = \begin{bmatrix} 0 & 0 \\ e_i B_i & e_i A_i \end{bmatrix}$.
8. Recursive call with input $(e_i, e_i A_i, e_i B_i)$ and output a list $[e_{ij}, \widetilde{A}_{ij}, \widetilde{B}_{ij}]_j$.
9. For each j , add to L : $\left[e_{ij}, \begin{bmatrix} 0 & 0 \\ \widetilde{B}_{ij} & \widetilde{A}_{ij} \end{bmatrix}, \begin{bmatrix} e_{ij} I & 0 \\ 0 & 0 \end{bmatrix} \right]$
10. If $e_i = 0$ for all i (L is still $[]$), output $[[e, A, B]]$, otherwise output L .

Example 11 We have used a symbolic software to compute canonical forms for rings $\mathbb{Z}/(d\mathbb{Z})$, where d is a squarefree integer. The following system over the ring $\mathbb{Z}/(210\mathbb{Z})$ splits into 3 subsystems:

$$\Sigma = \left(\begin{bmatrix} 21 & 158 & 169 & 147 \\ 138 & 208 & 43 & 135 \\ 67 & 46 & 190 & 100 \\ 167 & 36 & 81 & 203 \end{bmatrix}, \begin{bmatrix} 178 & 152 & 60 & 58 \\ 90 & 186 & 36 & 120 \\ 102 & 96 & 30 & 198 \\ 140 & 40 & 42 & 146 \end{bmatrix} \right)$$

$$105\Sigma = \left(\begin{bmatrix} 105 & 0 & 105 & 105 \\ 0 & 0 & 105 & 105 \\ 105 & 0 & 0 & 0 \\ 105 & 0 & 105 & 105 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right), 70\Sigma = \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 70 & 0 & 0 & 0 \\ 0 & 70 & 0 & 0 \\ 0 & 0 & 0 & 140 \end{bmatrix}, \begin{bmatrix} 70 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$36\Sigma = \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 36 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 36 & 0 & 0 & 0 \\ 0 & 36 & 0 & 0 \\ 0 & 0 & 36 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

We see that the classification of (A, B) reduces to the similarity classification of a 4×4 block of the first system, and the 1×1 block 140 appearing in the uncontrollable part of the second system, while the third system is reachable and completely described by

the sizes of the idempotent and zero blocks which determine its canonical form. E.g, the Brunovski form of 36Σ is

$$36 \cdot \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \right)$$

6 Concluding Remarks

Our results show that over a regular ring feedback classification of systems still behaves quite similar to the classical context over a field. But typically one has to handle a finite collection of systems at the same time. Each of these “parallel” systems can be transformed constructively into a strong Kalman decomposition, where the reachable part is in Brunovski canonical form and where the non reachable part can only be altered further via matrix similarity. Only if a normal form for similarity of matrices is known, as is the case over fields, our results give a complete normal form.

The connection of our normal form to the elementary divisors of the reachability submodule, which we state in Proposition 9 has not yet been worked out in the multi-input case.

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